

On Cannon-Thurston maps for relatively hyperbolic groups

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Abstract

Baker and Riley proved that a free group of rank 3 can be contained in a hyperbolic group as a subgroup for which the Cannon-Thurston map is not well-defined. By using their result, we show that the phenomenon occurs for not only a free group of rank 3 but also every non-elementary hyperbolic group. In fact it is shown that a similar phenomenon occurs for every non-elementary relatively hyperbolic group.

Keywords: Cannon-Thurston maps; relatively hyperbolic groups; geometrically finite convergence actions; convergence actions

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1 Introduction

Given an injective group homomorphism from a hyperbolic group to another hyperbolic group, whether the map can be continuously extended on the Gromov boundaries is an interesting question by Mitra (see [13, Section 1]). If such an extension is well-defined, the induced map on the Gromov boundaries is called the Cannon-Thurston map. The first non-trivial example was known by Cannon and Thurston in the 1980's (see [6]). Indeed their main theorem implies that for a closed hyperbolic 3-dimensional manifold M which fibers over the circle with fiber a closed hyperbolic surface S , when we consider the induced injective group homomorphism between fundamental groups of S and M , the Cannon-Thurston map is well-defined. Also more examples for which the Cannon-Thurston maps are well-defined can be recognized by Mitra's results

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(see [12] and [13]). At the present time, there are many works related to well-definedness of the Cannon-Thurston maps. Nevertheless Baker and Riley gave a negative answer ([2, Theorem 1]). Indeed they showed that a free group of rank 3 can be contained in a hyperbolic group as a subgroup for which the Cannon-Thurston map is not well-defined. In this paper we show that the phenomenon occurs for not only a free group of rank 3 but also every non-elementary hyperbolic group. In fact it is shown that a similar phenomenon occurs for every non-elementary relatively hyperbolic group.

Throughout this paper, every countable group is endowed with the discrete topology. We use a definition of relative hyperbolicity for groups from a dynamical viewpoint (see [5, Definition 1], [19, Theorem 0.1] and [9, Definition 3.1]). Also we use a definition of relative quasiconvexity for subgroups of relatively hyperbolic groups from a dynamical viewpoint (see [7, Definition 1.6]). Refer to [9, Section 3 and Section 6] for other several equivalent definitions of those. Also see [17], [4] and [5] for some definitions and properties related to convergence actions.

Let G be a non-elementary countable group and \mathfrak{H} be a conjugacy invariant collection of proper infinite subgroups of G . Suppose that G is hyperbolic relative to \mathfrak{H} , that is, there exists a compact metrizable space endowed with a geometrically finite convergence action of G such that \mathfrak{H} is the set of all maximal parabolic subgroups of G . Such a space is unique up to G -equivariant homeomorphisms and called the Bowditch boundary of (G, \mathfrak{H}) . In this paper we denote it by $\partial(G, \mathfrak{H})$. We remark that the set of conjugacy classes of elements of \mathfrak{H} is automatically finite by [18, Theorem 1B]. When the group G is hyperbolic, it is hyperbolic relative to the empty collection \emptyset and the Bowditch boundary $\partial(G, \emptyset)$ is nothing but the Gromov boundary ∂G .

We consider another non-elementary countable group G' which is hyperbolic relative to a conjugacy invariant collection \mathfrak{H}' of proper infinite subgroups of G' . Suppose that G is a subgroup of G' . Then we can consider the restricted action of G on $\partial(G', \mathfrak{H}')$ and the limit set $\Lambda(G, \partial(G', \mathfrak{H}'))$. If there exists a G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to $\partial(G', \mathfrak{H}')$, then it is unique and the image is equal to $\Lambda(G, \partial(G', \mathfrak{H}'))$ (see for example [11, Lemma 2.3 (1), (2)]). When the map exists, it is also called the Cannon-Thurston map. If the Cannon-Thurston map is well-defined, then any $H \in \mathfrak{H}$ is contained in some $H' \in \mathfrak{H}'$ (see for example [11, Lemma 2.3 (5)]). In general the converse is not true by [2, Theorem 1] (see also Lemma 2.1). Our main theorem claims that the converse is also not true for the case where the pair of G and \mathfrak{H} is not necessarily the pair of a free group of rank 3 and \emptyset . More precisely we have the following:

Theorem 1.1. Let G be a non-elementary countable group which is hyperbolic relative to a conjugacy invariant collection \mathfrak{H} of proper infinite subgroups of G . Then there exist a countable group G' containing G as a subgroup and a conjugacy invariant collection \mathfrak{H}' of proper infinite subgroups of G' satisfying the following:

- (i) the group G' is hyperbolic relative to \mathfrak{H}' ;

- (ii) every $H \in \mathfrak{H}$ belongs to \mathfrak{H}' and each $H' \in \mathfrak{H}'$ is conjugate to some $H \in \mathfrak{H}$ in G' ;
- (iii) there exists no G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to $\partial(G', \mathfrak{H}')$;
- (iv) the group G is not quasiconvex relative to \mathfrak{H}' in G' .

If we apply Theorem 1.1 for the case where G is hyperbolic and \mathfrak{H} is \emptyset , then G' is hyperbolic and \mathfrak{H}' is \emptyset . We remark that our proof uses [2, Theorem 1].

Remark 1.2. Theorem 1.1 (i), (ii) and (iv) can be considered as a generalization of [10, Theorem A] for relatively hyperbolic groups.

Let G be a countable group and X be a compact metrizable space endowed with a minimal non-elementary convergence action of G . We denote by $\mathfrak{H}(G, X)$ the set of all maximal parabolic subgroups with respect to the action of G on X and call it the peripheral structure with respect to the action of G on X . Let us consider another compact metrizable space Y endowed with a minimal non-elementary convergence action of G . When there exists a G -equivariant continuous map from X to Y , we say that X is a blow-up of Y and that Y is a blow-down of X . Suppose that the action of G on X is geometrically finite. [11, Proposition 1.6] claims that X has no proper blow-ups with the same peripheral structure. [11, Theorem 1.4] gives a family of uncountably infinitely many blow-downs of X with the same peripheral structure. On the other hand Theorem 1.1 implies that there exists a compact metrizable space endowed with a minimal non-elementary convergence action of G such that the peripheral structure is equal to $\mathfrak{H}(G, X)$ and it is not a blow-down of X . In fact the following is shown:

Corollary 1.3. Let G be a countable group. Let X be a compact metrizable space endowed with a geometrically finite convergence action of G . Then there exists a compact metrizable space Y endowed with a minimal non-elementary convergence action of G satisfying the following

- (i) $\mathfrak{H}(G, X) = \mathfrak{H}(G, Y)$;
- (ii) the spaces X and Y has no common blow-ups. In particular Y is not a blow-down of X .

Remark 1.4. For every non-elementary relatively hyperbolic group (resp. every non-elementary hyperbolic group), the second question (resp. the first question) in [14, Section 1] has a negative answer by this corollary. Also this corollary implies that every non-elementary relatively hyperbolic group does not have the universal convergence action which is defined by Gerasimov ([8, Subsection 2.4]).

2 Proof of Theorem 1.1

Before we show Theorem 1.1, we fix some notations. Let a countable group G act on a compact metrizable space X . Suppose that the action is a minimal

non-elementary convergence action. Then X can be regarded as a boundary of G . In fact $G \cup X$ has the unique topology such that this is a compactification of G and the natural action on $G \cup X$ is a convergence action whose limit set is X (see for example [11, Lemma 2.1]). Let L be a subgroup of G . Then the restricted action of L on X is a convergence group action. We denote by $\Lambda(L, X)$ the limit set. If L is neither virtually cyclic nor parabolic with respect to the action on X , then the induced action of L on $\Lambda(L, X)$ is also a minimal non-elementary convergence action.

We need the following lemma in order to show Theorem 1.1.

Lemma 2.1. Let G' be a countable group and have a subgroup G . Let X and X' be compact metrizable spaces endowed with minimal non-elementary convergence actions of G and G' , respectively. Then the following is equivalent:

- (i) there exists a G -equivariant continuous map from X to X' ;
- (ii) there exists a G -equivariant continuous map from X to $\Lambda(G, X')$;
- (iii) the injection $G \rightarrow G'$ is continuously extended to a map from $G \cup X$ to $G' \cup X'$.

Proof. The implication from (iii) to (i) (resp. from (i) to (ii)) is trivial. We show that (ii) implies (iii). Suppose that we have a G -equivariant continuous map ϕ from X to $\Lambda(G, X')$. Then this is extended to a continuous map $id_G \cup \phi : G \cup X \rightarrow G \cup \Lambda(G, X')$ (see [11, Lemma 2.3]). Since $G \cup \Lambda(G, X')$ is regarded as the closure of G -orbit of the unit element of G' in $G' \cup X'$, the injection $\iota : G \cup \Lambda(G, X') \rightarrow G' \cup X'$ is continuous. Then $\iota \circ (id_G \cup \phi) : G \cup X \rightarrow G' \cup X'$ is a continuous extension of the injection $G \rightarrow G'$. \square

Proof of Theorem 1.1. Since G has the maximal finite normal subgroup by [1, Lemma 3.3], we denote it by $M(G)$. By using [11, Theorem B.1], we take a subgroup $F' = F \times M(G)$ of G such that F is a free group of rank 3 and G is hyperbolic relative to

$$\mathfrak{H} \cup \{K \subset G \mid K = gF'g^{-1} \text{ for some } g \in G\}.$$

Take a hyperbolic group L containing F as a subgroup such that the injection $F \rightarrow L$ can not continuously extend on the Gromov-boundaries by [2, Theorem 1]. We remark that there exists no F -equivariant continuous map from ∂F to ∂L by Lemma 2.1. We put $L' := L \times M(G)$, $G' := G *_F L'$ and

$$\mathfrak{H}' := \{H' \subset G' \mid H' = g'Hg'^{-1} \text{ for some } H \in \mathfrak{H} \text{ and for some } g' \in G'\}.$$

By the construction, we have the condition (ii). Also it follows from [7, Theorem 0.1 (2)] that G' is hyperbolic relative to

$$\mathfrak{H}' \cup \{K \subset G' \mid K = g'L'g'^{-1} \text{ for some } g' \in G'\}.$$

Since L' is hyperbolic, we have the condition (i) by [15, Theorem 2.40].

Now we show the condition (iii). Assume that there exists a G -equivariant continuous map $\phi : \partial(G, \mathfrak{H}) \rightarrow \partial(G', \mathfrak{H}')$. The map ϕ implies F -equivariant continuous map $\Lambda(F, \partial(G, \mathfrak{H})) \rightarrow \Lambda(L, \partial(G', \mathfrak{H}'))$. Since F' (resp. L') is hyperbolically embedded into G (resp. G') relative to \mathfrak{H} (resp. \mathfrak{H}') in the sense of [16, Definition 1.4], F (resp. L) is strongly quasiconvex relative to \mathfrak{H} (resp. \mathfrak{H}') in G (resp. G') in the sense of [15, Definition 4.11] by [16, Theorem 1.5] and [15, Theorem 4.13]. Then the action on F (resp. L) on $\Lambda(F, \partial(G, \mathfrak{H}))$ (resp. $\Lambda(L, \partial(G', \mathfrak{H}'))$) is a geometrically finite convergence action without parabolic points (see [9, Theorem 9.9]). Hence $\Lambda(F, \partial(G, \mathfrak{H}))$ (resp. $\Lambda(L, \partial(G', \mathfrak{H}'))$) is F -equivariant (resp. L -equivariant) homeomorphic to the Gromov boundary ∂F (resp. ∂L) by [3, Theorem 0.1] and [18, Theorem 1A]. Hence ϕ gives an F -equivariant continuous map from ∂F to ∂L . This contradicts the fact that there exists no such maps.

Finally we show the condition (iv). Assume that G is quasiconvex relative to \mathfrak{H}' in G' . The peripheral structure with respect to the action of G on $\partial(G', \mathfrak{H}')$ is

$$\mathfrak{H}'' := \{P \subset G \mid P \text{ is infinite and } P = G \cap H' \text{ for some } H' \in \mathfrak{H}'\}.$$

By [7, Definition 1.6], $\partial(G, \mathfrak{H}'')$ is G -equivariant homeomorphic to $\Lambda(G, \partial(G', \mathfrak{H}'))$. Since we have $\mathfrak{H} \subset \mathfrak{H}''$ by the condition (ii), there exists a G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to $\partial(G, \mathfrak{H}'')$ by [11, Theorem 1.1]. Thus we have a G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to $\partial(G', \mathfrak{H}')$. This contradicts the condition (iii). \square

Proof of Corollary 1.3. For G and $\mathfrak{H} := \mathfrak{H}(G, X)$, we take G' and \mathfrak{H}' in Theorem 1.1 and put $Y := \Lambda(G, \partial(G', \mathfrak{H}'))$. Note that $X = \partial(G, \mathfrak{H})$ and $\mathfrak{H} = \mathfrak{H}(G, Y)$. Assume that there exists a common blow-up of X and Y . Since $\mathfrak{H}(G, X) = \mathfrak{H}(G, Y)$, we have a compact metrizable space Z endowed with a minimal non-elementary convergence action of G which is a common blow-up of X and Y such that $\mathfrak{H}(G, Z) = \mathfrak{H}(G, X) = \mathfrak{H}(G, Y)$ by [11, Lemma 2.6]. Then the G -equivariant continuous map from Z to X is a G -equivariant homeomorphism by [11, Proposition 1.6]. Hence Y is a blow-down of X and thus we have a G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to $\partial(G', \mathfrak{H}')$ by Lemma 2.1. This contradicts the condition (iii) in Theorem 1.1. \square

Remark 2.2. The space Y in the above proof cannot be written as inverse limit of any inverse system of compact metrizable spaces endowed with geometrically finite convergence actions of G (compare with [11, Theorem 1.4]). Indeed assume that Y is inverse limit of an inverse system of compact metrizable spaces X_i ($i \in I$) endowed with geometrically finite convergence actions of G . Since every element $H \in \mathfrak{H}$ is parabolic with respect to the action on Y and thus on X_i for each $i \in I$, there exists a unique G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to X_i for each $i \in I$ by [11, Theorem 1.1]. Hence we have a G -equivariant continuous map from $\partial(G, \mathfrak{H})$ to Y . This contradicts Corollary 1.3.

It may be interesting to ask whether a given compact metrizable space endowed with a geometrically infinite convergence action of G can be written as

inverse limit of some inverse system of compact metrizable spaces endowed with geometrically finite convergence actions of G .

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